Riemann Hypothesis: The Riesz-Hardy-Littlewood wave in the long wavelength region

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Abstract

We present the results of numerical experiments in connection with the Riesz and Hardy-Littlewood criteria for the truth of the Riemann Hypothesis (RH). The coefficients c_k of the Pochammer's expansion for the reciprocal of the Riemann Zeta function, as well as the "critical functions" $c_k k^a$ (where a is some constant), are analyzed at relatively large values of k. It appears an oscillatory behaviour (Riesz-Hardy-Littlewood wave). The amplitudes and the wavelength of the wave are compared with an analytical treatment concerning the wave in the asymptotic region. The agreement is satisfactory. We then find numerically that in the large β limit too, the amplitudes of the waves appear to be bounded. For a special case the numerical experiments are performed up to larger values of k, i.e $k = 10^9$ and more. The analysis suggests that RH may barely be true and an absolute bound for the amplitudes of the waves in all cases should be given by $|\frac{1}{\zeta(\frac{1}{2}+\epsilon)} - 1|$, with ϵ arbitrarily small positive, i.e. equal to $1.68\ldots$

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I. INTRODUCTION

Following recent works concerning the study of some well known functions appearing in the original criteria of Riesz, Hardy and Littlewood for the possible truth of the Riemann Hypothesis (RH), there is new interest in the direction of numerical experiments, where the calculations use the ideas of some recent works on the subject. These concern the expansion of the reciprocal of the Riemann Zeta function in terms of the so called Pochammer's polynomials P_k, whose coefficients c_k play a central role also in the asymptotic region of very large k [1, 2, 3, 4]. For new zero free regions of the Zeta function, in the context of a rigorous treatment with the Müntz formula, the reader may consult a recent work by Albeverio and Cebulla [5].

Here we are concerned with the discrete version of the Riesz criterion which has also been studied numerically: the first numerical experiments for values of k up to 100'000 have been announced and reported for the Riesz case in [2, 6]. It has been found that the function c_k has a oscillatory behaviour in a region of relatively high k's, in agreement with an asymptotic formula given by Baez-Duarte [2]. The agreement appears satisfactory even if only the contribution of the first non trivial zero of the Zeta function located at $s = \frac{1}{2} + i14.134725$ in the complex plane has been used.

In a previous work [7] a two parameter family (parameter α and β) of Pochammer's polynomials was introduced. This allowed the starting investigation of c_k at low values of k, but in various cases and in the so called "strong coupling" regime (high β). After the initial study at low k, our computations using the formula containing the Möbius function were easily extended to larger and larger k (up to a billion) in the strong coupling limit, with the appearance of macro-oscillations in c_k extending to larger k. This is a symptom that using such a limit the RH may eventually barely be true[7]. In this work we continue the numerical experiments also using our Poisson formula established in [7] which is well suited for numerical purposes.

After the formulation of the model in Section 2, we then compute in Section 3 the amplitudes of what we call the Riesz-Hardy-Littlewood wave, which contains arbitrarily scales in few of the two parameters α and β at our disposal. Using the Baez-Duarte formula, we then present our results for different models up to values of k equal to one billion and observe oscillations in all cases (Section 4). The agreement with the asymptotic formula of

Baez-Duarte is satisfactory. Then, in Section 5, we concentrate the study in more details by considering a special new model already proposed in [7] where $\alpha = \frac{7}{2}$ and β is increasing starting with the value equal to 4. The results show in a concrete way the "transition" from the low coupling to the "strong coupling regime": at low values of β ($\beta = 4$) we obtain up to 7 oscillation with values of k extending up to a billion. These start to deform continuously with increasing values of β approaching the infinite β limit. In such a regime, the wave is absorbed in a macroscopic region with an amplitude whose strength should be finite as already noted in [7].

In the context of validity of our numerical results, our analysis gives further indication that the RH may barely be true as indicated by our two parameter models in the week as well as in the "strong coupling regime" (Section 5). Moreover, the possibility that in an ideal numerical experiment (using an arbitrarily large but finite maximum value of n, say N in the formula with the Möbius function) the amplitude of the waves at finite β values should decrease, is commented in Appendix.

II. THE MODEL

Following recent treatments[1, 2, 7], a possible expansion of the reciprocal of the Zeta function i.e. $\zeta(s)^{-1}$ in terms of the so called Pochammer's polynomials P_k, with two parameters α ($\alpha > 1$) and β is this one:

$$\frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k(\alpha, \beta) P_k(s, \alpha, \beta) \tag{1}$$

where

$$P_k(s, \alpha, \beta) = \prod_{r=1}^{k} \left(1 - \frac{\frac{s-\alpha}{\beta} + 1}{r} \right) \tag{2}$$

$$c_k(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} \left(1 - \frac{1}{n^\beta} \right)^k \tag{3}$$

and $P_k(0, \alpha, \beta) = 1$.

In (3) the Möbius function of argument n is given by:

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0, & \text{if } n \text{ contains a square} \end{cases}$$

If $s = \rho + it$ is a complex variable and argument of the Riemann Zeta function one has for $\Re(s) = \rho > 1$:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{4}$$

Another explicit formula for the $c_k(\alpha, \beta)$ is obtained from (3) using the binomial coefficients and reads:

$$c_k(\alpha, \beta) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(\alpha + \beta j)} \tag{5}$$

As β is increasing, one may also use (especially) in the context of numerical experiments, the formula recently obtained [7] and given by:

$$c_k(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^\alpha} e^{-\frac{k}{n^\beta}} \tag{6}$$

In such an approximation we have that

$$c_k(\alpha, \beta) = \sum_{p=0}^{\infty} c_p(\alpha, \beta) \frac{k^p}{p!} e^{-k} \tag{7}$$

which shows the emergence of a Poisson like distribution for the coefficients $c_k(\alpha, \beta)$. This should be a very satisfactory approximation in the limit of relative large values of β [7]. We recall that an important inequality due to Baez-Duarte [2], concerning the Pochammer's polynomials of complex argument z is given by:

$$|P_k(z)| \leq C k^{-\Re(z)} \tag{8}$$

The above inequality applied to our two parameter family of Pochammer's polynomials with complex argument $z = \frac{s-\alpha}{\beta} + 1$ gives:

$$|P_k(s, \alpha, \beta)| \leq C k^{-\frac{\rho-\alpha}{\beta}+1} \tag{9}$$

So that $\zeta(s)$ in (1) will be different from zero and thus the RH will be true for $\Re(s) = \rho > \frac{1}{2}$ if the c_k decay, at large k, as (see [7]):

$$|c_k| \leq A k^{-\frac{\alpha-\rho}{\beta}} \tag{10}$$

We will also consider the "critical function"

$$\psi(k; \alpha, \beta, \rho) := c_k k^{\frac{\alpha-\rho}{\beta}} \tag{11}$$

TABLE I: The expected decay of c_k for different values of α and β

α	β	ρ	decay of $\lvert c_k\rvert$	Note
2	2	$\frac{1}{2}$	$k^{-\frac{3}{4}}$	The case of Riesz
1	2	$\frac{1}{2}$	$k^{-\frac{1}{4}}$	The case of Hardy-Littlewood
2	6	$\frac{1}{2}$	$k^{-\frac{1}{4}}$	Same decay as the Hardy-Littlewood case but numerically more convenient
$\frac{7}{2}$	4	$\frac{1}{2}$	$k^{-\frac{3}{4}}$	Same decay as the Riesz case, intensive calculations are given below
3	3	$\frac{1}{2}$	$k^{-\frac{5}{6}}$	If the Zeta function has no zero for $\rho > \frac{3}{4}$ then $c_k(3,3)$ should decay at least as $k^{-\frac{3}{4}}$
4	4	$\frac{1}{2}$	$k^{-\frac{7}{8}}$	Since from the Prime number theorem there is no zero for $\rho > 1$
				the $c_k(4,4)$ decays at least as $k^{-\frac{3}{4}}$
2	4	$\frac{1}{2}$	$k^{-\frac{3}{8}}$	Another interesting case for calculations

which from (10) is expected to be bounded by a constant A.

We now recall two original cases given in pionnering works by Riesz [8] and by Hardy-Littlewood [9]. Setting $\rho = \frac{1}{2}$ in (10), for $\alpha = \beta = 2$ (Riesz case) we have that $|c_k| \leq Ak^{-\frac{3}{4}}$ and for $\alpha = 1$, $\beta = 2$ (Hardy-Littlewood case) $|c_k| \leq Ak^{-\frac{1}{4}}$. Other interesting cases for which we will carry out intensive numerical experiments to be presented below are summarized in the Table I.

A limiting delicate case analyzed in [7] is the one where $\alpha = \frac{1}{2} + \delta$ and β grows to infinity. Here of course we do not have absolute convergence to $\zeta(s)^{-1}$ (c_k may nevertheless be analyzed) and from (10) we have that the c_k should be smaller then a constant for all k. This is what we verified with numerical experiments (not presented here) with values of k up to a billion. The value of the constant has been proposed in our previous work [7] and the conjecture was that $|c_k| \leq |\frac{1}{\zeta(\frac{1}{2})} - 1| \cong 1.68477$. However the situation is delicate ($\alpha < 1$) since Littlewood [10] has shown that, assuming RH is true, $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\frac{1}{2}+\epsilon}}$ is convergent for all ϵ strictly greater than zero.

The general situation is that the "critical function" $c_k k^{\frac{\alpha-\rho}{\beta}}$ should be bounded by a constant in absolute value as $k \to \infty$. In fact the function starts at zero for $k = 0$, reaches a minimum, then starts to increase and then begins to oscillate with a "constant " amplitude as $k \to \infty$ as we will see in the experiments. In a previous work [7] we have analyzed c_k in various cases but only for moderately values of k, i.e for k not exceeding 1000, with exception of some cases at large values of β, where k reached the value of a half billion. c_k was found

to have only negative values in the range considered and increasing with k. Presently we know of recent numerical experiments [6] in the Riesz case carried out by Maslanka (k up to 100'000) and Wolf (k up to 200'000). These calculations show that c_k become of oscillatory type, thus assuming positive and negative values with an amplitude which appears constant in the range considered. In fact two or three oscillations with a wavelength related in first approximation to the first zero of the Riemann Zeta function may be seen. Here it should be remarked that this situation for the Riesz case is not in contraddiction with our strong coupling limit (β large) cited above (see discussion below for the case $\alpha = \frac{7}{2}$ and β increasing).

In few of these new finding, we want first analyze (in an analytical context) such a behaviour and we call this general phenomena the Riesz-Hardy-Littlewood wave. This will be analyzed using an interesting result of Baez-Duarte, i.e. an expression giving c_k for $k \to \infty$.

III. THE RIESZ-HARDY-LITTLEWOOD WAVE

For the Riesz case, in connection with the Mellin inversion formula, the Riesz function is given (see [8] and [11]) explicitly by:

$$F(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\zeta(2k)} \tag{12}$$

Using the calculus of residues *F(x)* is obtained by an integration and is given by:

$$F(x) = \frac{i}{2\pi} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(1-s)x^s}{\zeta(2s)} ds \tag{13}$$

where $\frac{1}{2} < a < 1$.

Now, recently Baez-Duarte [2], with an ingenious method found in particular an expression for the reciprocal of the Pochammer polynomial given by:

$$\frac{1}{P_k(s)} = \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{j}{s-j} \tag{14}$$

where uniformely on compact subsets one has:

$$\lim_{k \to \infty} P_k(s) k^s = \frac{1}{\Gamma(1-s)} \tag{15}$$

and he was able to obtain an explicit formula connecting c_k and the set of all trivial and non trivial zeros (z denote the complex Zeta zeros, $z = \frac{1}{2} + it$) under the assumption of simple zeros. For the Riesz case the expression is given by:

$$-2kc_{k-1} = \sum_{\Im(z)} \frac{1}{\zeta'(z)P_k(\frac{z}{2})} \tag{16}$$

for sufficiently large k. It should be said that formula (16) of Baez-Duarte is very nice and may be used to control our numerical computations at large k to be presented below. Apparently (16), with some precautions, may be extended to the general case with parameters α, β and should read:

$$-\beta kc_{k-1} = \sum_{\Im(z)} \frac{1}{\zeta'(z)P_k(\frac{s-\alpha}{\beta} + 1)} \tag{17}$$

To obtain an asymptotic value for the amplitude of the Riesz-Hardy-Littlewood wave, we use (15) in (17):

$$-\beta kc_{k-1} = \sum_{\Im(z)} \frac{k^{\frac{it}{\beta}} k^{\frac{\frac{1}{2}-\alpha}{\beta}}}{\zeta'(z)} \tag{18}$$

where $t = \Im(z)$. In the limit of large k, one may neglect the contribution of the trivial zeros [2]. For the "critical function" we then have the following expression:

$$k^{\frac{\alpha-\frac{1}{2}}{\beta}} c_k \cong -\frac{1}{\beta} \sum_{\Im(z)} \frac{k^{\frac{it}{\beta}} \Gamma(-\frac{\frac{1}{2}+it-\alpha}{\beta})}{\zeta'(z)} =: \overline{\psi}(k; \alpha, \beta, \frac{1}{2}) \tag{19}$$

for large k. To prepare the comparison of (19) with the numerical results we write explicitly (19) for the various cases we will treat. In order to obtain an estimate for the amplitude of the wave in the long wavelength limit (k large) we will use here only the first zero of the Riemann Zeta function up to 10 decimals (β small).

The upper bounds for the amplitude of the waves above, will be compared with the results of the numerical experiments performed for the various cases using (3).

IV. NUMERICAL EXPERIMENTS

We now present the results of our numerical experiments which was carried out in more cases using the Möbius function in (3) up to $n = 10^6$. We calculated c_k until $k = 10^6$ or $k = 10^9$ with a scaling factor of 2500 for the k-axis. These will be compared with the

TABLE II: The "amplitude" of ψ for different values of α and β

α	β	The function	The amplitude
2	2	$\lvert k^{\frac{3}{4}}c_k\rvert = \lvert\psi(k;2,2,\frac{1}{2})\rvert$	0.000078
1	2	$\lvert k^{\frac{1}{4}}c_k\rvert = \lvert\psi(k;1,2,\frac{1}{2})\rvert$	0.0000292558
2	6	$\lvert k^{\frac{1}{4}}c_k\rvert = \lvert\psi(k;2,6,\frac{1}{2})\rvert$	0.0210433
$\frac{7}{2}$	4	$\lvert k^{\frac{3}{4}}c_k\rvert = \lvert\psi(k;\frac{7}{2},4,\frac{1}{2})\rvert$	0.008411
3	3	$\lvert k^{\frac{5}{6}}c_k\rvert = \lvert\psi(k;3,3,\frac{1}{2})\rvert$	0.0021562
4	4	$\lvert k^{\frac{7}{8}}c_k\rvert = \lvert\psi(k;4,4,\frac{1}{2})\rvert$	0.00984936
2	4	$\lvert k^{\frac{3}{8}}c_k\rvert = \lvert\psi(k;2,4,\frac{1}{2})\rvert$	0.0052445

upper bound for the amplitude of the waves of Section 3. The general situation is that for moderately values of k (until some tausend) the wave given by the experimental results start with zero amplitude, after a minimum with a negative value, increases and seems to stabilize at large values of k with oscillations displaced at larger and larger wavelength (proportional to $\log(k)$) and with an amplitude which seems to saturate to a constant value (given in a good approximation) by the upper bound (19). Below (Figure 1) we first give the plots of the wave for the Riesz case ($\alpha = \beta = 2$). As remarked in [2], the first intensive calculations with very high precision up to $k = 100'000$ (by K. Maslanka) and up to $k = 200'000$ (by M. Wolf) indicated the appearance of oscillations with the first one in the region $k = 20'000$. Our results obtained with (3) confirm for such values the asymptotic limit for the wave with an amplitude in agreement with the bound obtained above ($A \cong 0.000078$). Notice that the minimun for low k values is 0.4 in absolute value as found in a previous work, is much bigger then A, which concerns only the asymptotic region of the wave, thus no disagreement!

Figure 2 and Figure 3 concern two cases of special interest since the decays are expected to be the same as for the Hardy-Littlewood case and for the Riesz case. In both cases there is agreement with the bound $A \cong 0.0210433$ and $A \cong 0.008411$ given above but the amplitudes are respectively 1000 and 100 time bigger than in the former cases.

In Figure 4 and Figure 5 we give the plots of $k^{\frac{5}{6}}c_k(3,3)$ and $k^{\frac{7}{8}}c_k(4,4)$ where the amplitudes are found to be in agreement with the theoretical upper bounds given above in Table 2, too.

The next special case is the one with $\alpha = 2$ and $\beta = 4$. Again, the experimentally

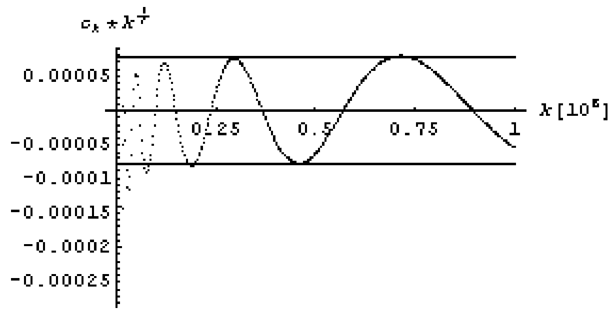

FIG. 1: The wave $k^{\frac{3}{4}}c_k$ for the Riesz case $\alpha = \beta = 2$

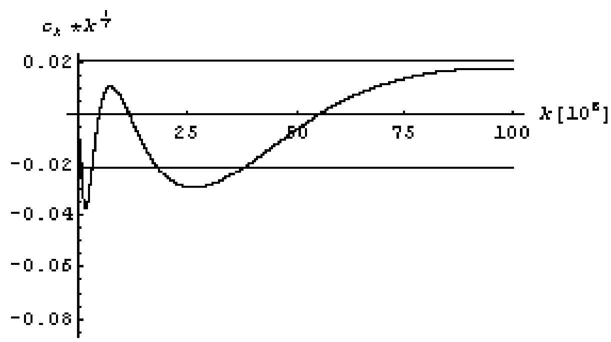

FIG. 2: The wave $k^{\frac{1}{4}}c_k$ for the case $\alpha = 2, \beta = 6$

detected amplitude agrees well with the theoretical bound given above, i.e $A = 0.0052445$.

As a further illustration we compare the wave $k^{\frac{3}{8}}c_k$ with the asymptotic approximation wave given by the wave $\psi(k; 2, 4, \frac{1}{2})$ (case $\alpha = 2, \beta = 4$). In the range for k from $1 \cdot 10^6$ to $10 \cdot 10^6$ the two waves appear to be walking close together arm in arm (Figure 7). Notice that in the approximation we considered only the contribution of the first zero given by $z = \frac{1}{2} + i14.134725141$ which appears dominant for low values of β.

V. THE CASE $\alpha = \frac{7}{2}$ AND β INCREASING

For $\alpha = \frac{7}{2}$ we will now present the plots of the waves for an increasing sequence of β values i.e. 4, 8, 12, and 20 (in order to investigate the "infinite beta limit" already introduced

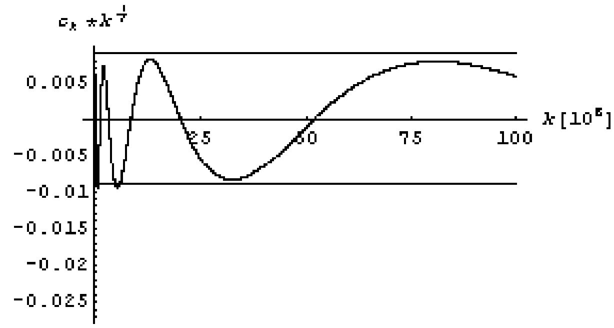

FIG. 3: The wave $k^{\frac{3}{4}} c_k$ for the case $\alpha = \frac{7}{2}, \beta = 4$

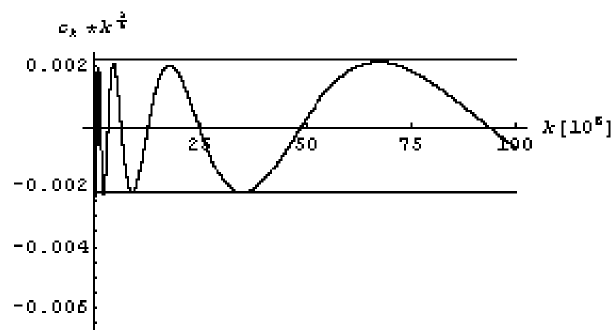

FIG. 4: The wave $k^{\frac{5}{6}} c_k$ for the case $\alpha = \beta = 3$

in our previous work [7]). We will compute the function

$$\psi(k; \frac{7}{2}, \beta, \frac{1}{2}) = k^{\frac{3}{\beta}} c_k(\frac{7}{2}, \beta) \tag{20}$$

which will also be compared with the expression given by the Baez-Duarte formula (19) in the asymptotic region $k \to \infty$. Here we will take into account only the contribution of the groundstate of the spectrum i.e $z = \frac{1}{2} + i14.134725141$. It is then convenient to introduce the new variable $x = \log(k)$. This allow us to control more efficiently the wavelength and the amplitude of the wave in the region to be considered (x runs from 8 to 22, so k up to $3.6 \cdot 10^9$).

In the Figures 8-11 we present our numerical results for increasing β values, which we call the "strong coupling limit".

At the same time it is seen that in this case $|c_k|$ itself is smaller than (c_k is not the critical

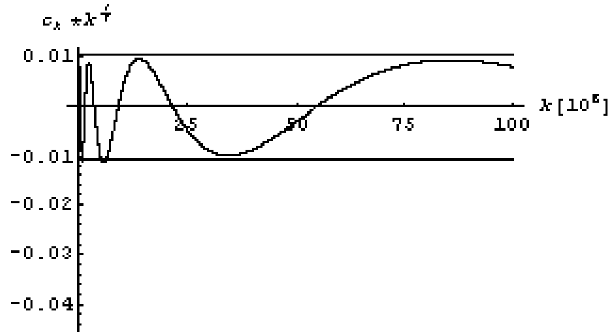

FIG. 5: The wave $k^{\frac{7}{8}} c_k$ for the case $\alpha = \beta = 4$

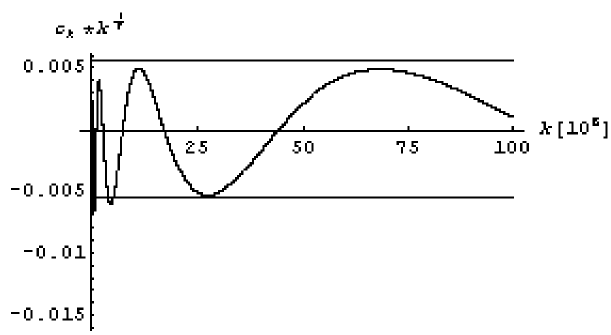

FIG. 6: The wave $k^{\frac{3}{8}} c_k$ for the case $\alpha = 2, \beta = 4$

function!):

$$\left| \frac{1}{\zeta(\frac{7}{2})} - 1 \right| \cong 0.11247897 \tag{21}$$

at least for the case $\beta = 4$ as already discussed in our previous work [7] concerning only very low values of k. Figure 12 confirm this behaviour also for large value of k. For this example the region of annihilation of the “eincoming“ wave extends up to larger and larger values of k. It should be noted that for the critical function ψ (20) the situation is more delicate since the value of a possible bound on ψ depends on β as it is been from our numerical results.

VI. CONCLUSIONS

In this work we have extended the numerical experiments obtained in our previous work and considered many cases of “waves” (with two parameters α and β). The experiments allow

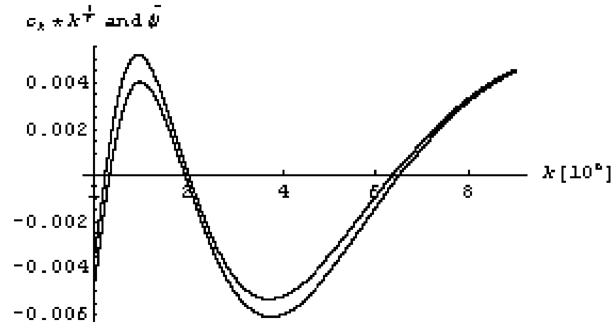

FIG. 7: The wave $k^{\frac{3}{8}} c_k$ (lowest curve) and the approximation $\overline{\psi}$ (highest curve)

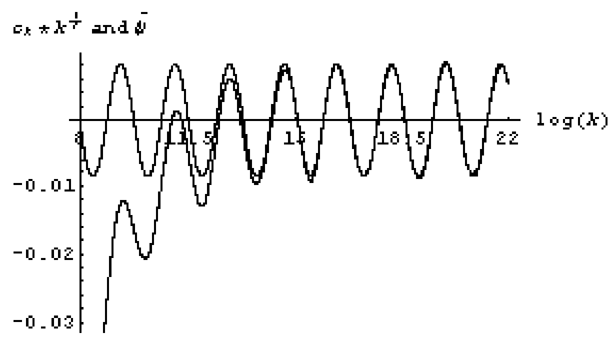

FIG. 8: The wave $k^{\frac{3}{4}} c_k$ (lowest curve) and the approximation $\overline{\psi}$ (highest curve), $\beta = 4$

to obtain up to 6 oscillations at low β values for (20) whose amplitudes are in agreement with those given by an extension of an asymptotic formula due to Baez-Duarte for the case $\alpha = \beta = 2$.

In the process of increasing the values of β (at least in the case $\alpha = \frac{7}{2}$) it has been shown that the width of the "annihilation" region increase with β where the amplitude of the wave seems (still) to remain bounded. In this connection we may argue that the results of our numerical experiments indicate that RH may barely be true due to the behaviour of ψ in the large β limit. It is also conjectured that an absolute bound on ψ, for all α and β allowed, should be given by $|\frac{1}{\zeta(\frac{1}{2}+\epsilon)} - 1| \cong 1.68$. In fact Littlewood has shown that on RH, $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\frac{1}{2}+\epsilon}}$ is convergent for all $\epsilon > 0$. In the light of the results of the experiments obtained so far and with the understanding that our remark is speculative, we belive that even with more sofisticated experiments it will be very hard to obtain values of the critical function

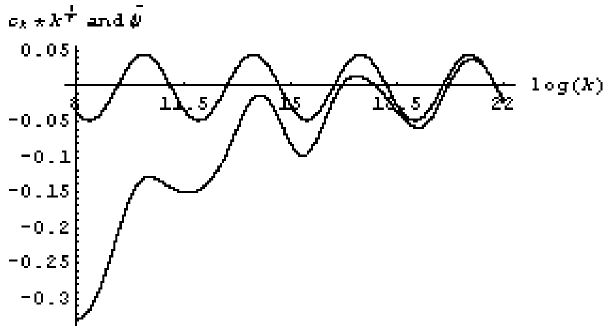

FIG. 9: The wave $k^{\frac{3}{8}}c_k$ (lowest curve) and the approximation $\overline{\psi}$ (highest curve), $\beta = 8$

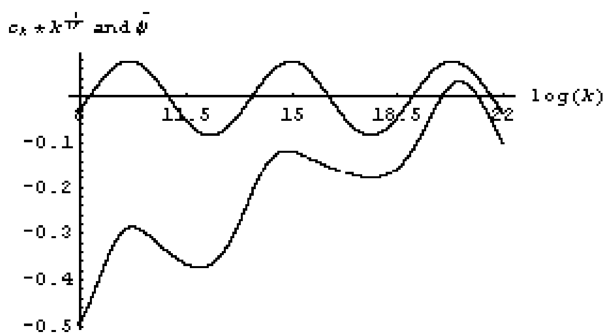

FIG. 10: The wave $k^{\frac{3}{12}}c_k$ (lowest curve) and the approximation $\overline{\psi}$ (highest curve), $\beta = 12$

$k^a c_k$ which in absolute value will be greater at large k then those of the infinite β limit, as commented in the Appendix.

APPENDIX A

In the context of the numerical experiments performed so far, it is helpful to obtain a crude inequality concerning a bound on the critical function. This is simply obtained by setting $|\mu(n)| = 1$ in (3). The critical function in the representation of $\frac{1}{\zeta(s)}$ in terms of the two parameter Pochammer's polynomials is given by:

$$k^{\frac{\alpha-\frac{1}{2}}{\beta}} c_k \cong k^{\frac{\alpha-\frac{1}{2}}{\beta}} \sum_{n=1}^{N} \frac{\mu(n)}{n^{\alpha}} \left(1 - \frac{1}{n^{\beta}}\right)^{k} =: f_k(\alpha, \beta, N)$$

where N is the maximum value of the argument in the Möbius function considered in a ideal

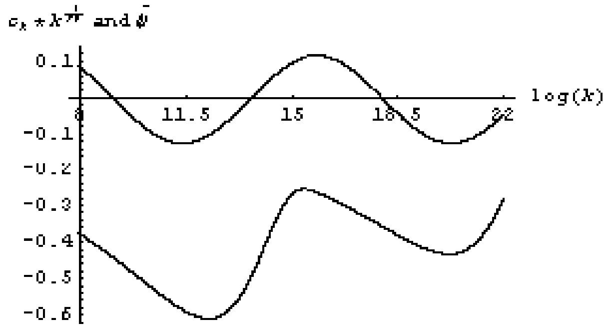

FIG. 11: The wave $k^{\frac{3}{20}} c_k$ (lowest curve) and the approximation $\overline{\psi}$ (highest curve), $\beta = 20$

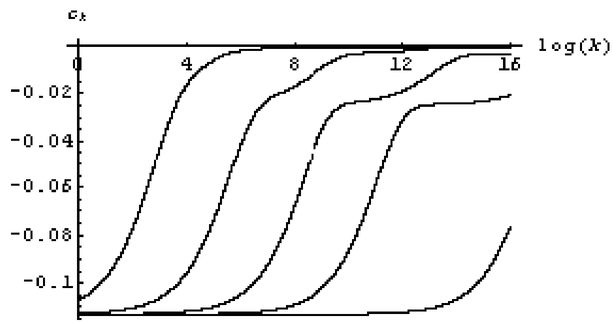

FIG. 12: $c_k(\frac{7}{2}, \beta)$ for $\beta = 4, 8, 12, 16, 24$ (from left to right)

numerical experiment (N finite). We then have, introducing the variable $x = \log(k)$ that:

$$|f_k(\alpha, \beta, N)| \leq e^{\frac{\alpha - \frac{1}{2}}{\beta} x} (\zeta(\alpha) - 1) e^{\log(1 - \frac{1}{N\beta}) e^x}$$

For large N we have:

$$|f_k(\alpha, \beta, N)| \leq (\zeta(\alpha) - 1)\, e^{\frac{\alpha - \frac{1}{2}}{\beta} x - \frac{1}{N\beta} e^x}$$

As an example we consider our case $\alpha = \frac{7}{2}$ and $\beta = 4$. Remembering that from Table 2 the amplitude calculated only with the first non trivial zero is about 0.008411, we may ask: for what N and k, $|f_k(\alpha, \beta, N)|$ is bounded by the value 0.008411? For example the inequality is satisfied for the followig pairs:

$$N = 1000 \quad x > 31$$
$$N = 10^6 \quad x > 60$$

As a second example we consider the Riesz case ($\alpha = \beta = 2$). From the Table 2, the amplitude (still restricting to the contribution of the first zero) is 0.000078. The inequality is satisfied as follows:

$$N = 1000 \quad x > 17$$
$$N = 10^6 \quad x > 31$$
$$N = 10^9 \quad x > 87.2$$

Now, still for the Riesz case

$$\left|\frac{1}{\zeta(s)}\right| \leq \sum_{k=0}^{\infty} |P_k(s,2,2)c_k(2,2)|$$

assuming as seen in the experiments that $k^{\frac{3}{2\beta}}c_k(2,\beta)$, i.e. the critical function is bounded then we would have that:

$$\left|\frac{1}{\zeta(s)}\right| \leq \sum_{k=0}^{\infty} \frac{1}{k^{1+\frac{\delta}{2}}} k^{\frac{3}{2\beta}} c_k(2,\beta) \leq C$$

Thus, $\frac{1}{\zeta(s)}$ would be different from zero for $\Re(s) > \frac{1}{2} + \delta, \delta > 0$. It should be said that for $\alpha = \frac{3}{2}$ and $\beta = 1$, in the numerical experiments, the wave seems to decay at zero after a few of oscillations, a situation very different from the case $\alpha = \frac{7}{2}$ and $\beta = 4$ (but the above inequality apply also in this case).

So, if the critical function for any β is in absolute values bounded for $k > K$ by the value of the infinite β limit 1.68..., the RH should be true.

In the plot of ψ for the case $\alpha = \frac{7}{2}$ and $\beta = 4$ (Figure 13) up to $k = 10$ billions we see 8 oscillations.

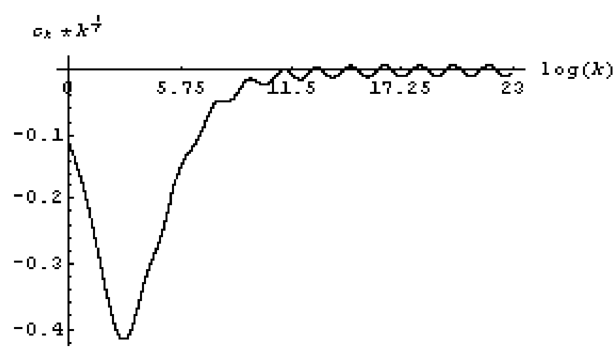

FIG. 13: $k^{\frac{3}{4}}c_k$ for $\alpha = \frac{7}{2}$ and $\beta = 4$ until $k = 10 \cdot 10^9$

Final comments.

1. Further application of the crude inequality considered in this Appendix indicates that in a ideal experiment using (3), with $N = 10^{10}$, the amplitude of the wave for $\log(k) < 23$ will change at most 10^{-6} time the value 0.008411... obtained with $N = 10^6$ in (3) and compatible with the Baez-Duarte amplitude using only the first nontrivial zero of Zeta. This indicates some stability of the numerical experiments in the intermediate range $\log(k) < 23$ (see Figure 13).

2. In the general case of an ideal experiment with large N in (3) the crude inequality indicates also that for $k > K(N)$, K finite, the amplitude of the critical function ψ is bounded in absolute value by $|\frac{1}{\zeta(\frac{1}{2})} - 1| \cong 1.68$, choosing N as needed. In the same way we may argue that the amplitude of the critical function may be obtained as small as we want (in particular smaller then 0.008411...) choosing N as needed for values of $k > K(N)$, K still finite.

3. One of the open questions is now the following: the critical function at large value of k is growing, stabilizing to a "periodic pure wave" with constant amplitude or decaying with a zero amplitude? From the results of our numerical treatment we are more in favour of the last two cases.

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